

## **How does complex mathematical theory arise? Phylogenetic and cultural origins of algebra**

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Algebra is a form of human cognition with specific emergent properties that are not found either in the cultural context in which mathematicians work, or in the evolved cognitive abilities for mathematical thought that enable it. In this paper, I examine the development of algebra in disparate historical cultural settings. I argue that the development of algebra relies on the interplay of cognitive and cultural factors, including a metaphysical view of the world compatible with science, a notation system that enables the symbolic notation of operations, and an ontological standpoint that perceives mathematics as a human endeavour. I begin by outlining evidence for mathematical knowledge in animals and infants. I show that without cultural evolution, mathematical knowledge remains simple and shallow. Then, using algebra as a case-study, I demonstrate how cumulative cultural evolution allows humans to partly overcome these cognitive limitations by combining several cognitive systems. Finally, I explain how many open complex systems, including religion, symbolic notation, and philosophical factors influence the development and level of abstraction in these algebraic systems in China, the Islamic world and Europe.

### **1. Mathematics without culture**

Even a concept as seemingly simple as the number two is a highly abstract representation. There is no obvious perceptual similarity between two cows and two bicycles, and yet both collections share the concept of 'twoness'. A large number of experimental studies nevertheless indicate that both non-human animals and human infants have a robust understanding of number (see De Cruz 2005 for an overview). For example, when presented with two different numerosities of food-items, both rhesus monkeys (Hauser et al. 2000) and red-backed salamanders (Uller et al. 2003) go for the larger quantity. Human infants, likewise, can solve simple addition and subtraction problems with both small and larger quantities (Wynn 1992, McCrink & Wynn 2004). Recently, single cell recordings in rhesus monkeys (Nieder et al. 2002, Nieder & Miller 2003) have identified number-sensitive neurons (Fig. 1). Each neuron is tuned to a preferred quantity: a neuron preferentially firing at, say 2, will fire a bit less at one or three, and even less when observing higher quantities. These neural tuning curves become broader as quantities increase. Numerosities are thus not represented as a linear mental number line, but more as a logarithmic ruler, in which the psychological distance between one and two is much greater than between 50 and 100. From an evolutionary perspective, this seeming fuzziness may actually prove to be more efficient than a linear, exact representation of number. After all, the difference between one apple and two

apples is more ecologically significant than between 50 apples or a hundred apples (Opfer 2005).

Yet, a mathematical system without exact number representations beyond three can only capture the most rudimentary of numerical relationships. Evidence for this claim comes from two Amazonian cultures, the Pirahã (Gordon 2004) and the Mundurukú (Pica et al. 2004), where exact number words do not exist, such that the word for ‘one’ can mean ‘two’ and vice versa. People in these cultures cannot even discriminate a box with three fish from a box with four fish painted on it. They do not possess a counting routine or any other cultural tool which helps them to construct positive integer values.

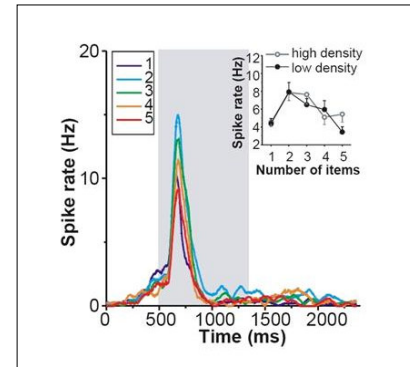


Fig. 1. number-sensitive neuron (from Nieder et al. 2002, 1710)

## 2. Cognitive processes involved in the transmission of algebra

I now turn to the case of classical algebra, sets of procedures and rules for solving equations with one or more unknowns. How do individual people acquire this procedural knowledge? It is important to note that the transmission of a concept such as ‘capitalism’ does not entail that this concept is downloaded from one brain to another: when I hear about capitalism, I reconstruct it in my own mind, and it is quite likely that my or your concept of capitalism is not the same (Sperber 1996). Some concepts are more easily transmitted than others because they require minimal cultural input – in such cases the rest of the concept is ‘filled in’ by the recipient’s tacit assumptions. For example, someone who learns that a zombie does not move according to his own free will tacitly assumes that such a being does obey the laws of physics. Thus, only the counterintuitive part of this concept – a violation of intuitions about agency – needs to be transmitted (Boyer 2000). Given the minimal transmission requirements for such religious concepts, it is not surprising that they should have evolved many times independently, occurring throughout a wide array of physical and cultural settings. Neither their birth nor persistence seems to require exceptional cultural conditions (McCauley 2000).

In stark contrast to religious concepts, algebra is *non-intuitive*: it requires a co-optation of evolved cognitive strategies (Geary 1995) which are normally not deployed together. This results in a new cognitive system with emergent properties not present in any of the cognitive subsystems. We can observe this in other domains as well: hunter-gatherers for instance, track down animals using a combination of several domains of knowledge outside of biology, including physics (like examining spoor and damage the animal caused to plants) and psychology (imaging what one would do if one were in the position of the prey) (Carruthers 2002). As a result, the emergent properties of tracking enable the tracker to work out solutions which would have been impossible in each of the subsystems. The drawback for this enhanced computational power is that such skills are typically hard to learn. Male hunter-gatherers start learning to track when they are adolescents but only gain expertise – as measured by their return-rates - when they are well in their thirties (Kaplan et al. 2000).

Likewise, fMRI studies (e.g. Qin et al. 2004) found that people use different brain circuits when they solve equations: the intraparietal sulci, which have been implicated in brain imaging studies involving number (e.g. Dehaene et al. 1999).; the anterior cingulate cortex, which is otherwise active when people reflect on other people’s mental states (Frith & Frith 2001); and the posterior parietal cortex, which is normally activated in visuospatial tasks, including spatial working memory and attention orienting (Simon et al. 2002) (Fig. 2).

One study investigating the difference between mathematical strategies in normal and in precocious teenagers found that the latter deployed spatial skills to solve algebraic problems by diagramming important relationships apparent in the problem (Dark & Benbow 1991). Thus, solving equations depends on a successful co-optation of several brain areas which are normally involved in ecologically relevant tasks such as detecting numerosities, and assessing spatial relationships.

Perhaps the most surprising finding is that adolescents exhibited alterations in their cerebral blood flow when solving equations after a learning period (Luna 2004). Algebra as cultural construction is clearly more than the sum of the specialized cognitive subsystems that create and sustain it, since its learning alters synaptic connections. I will now look at the role of the cultural context in the development of algebra.

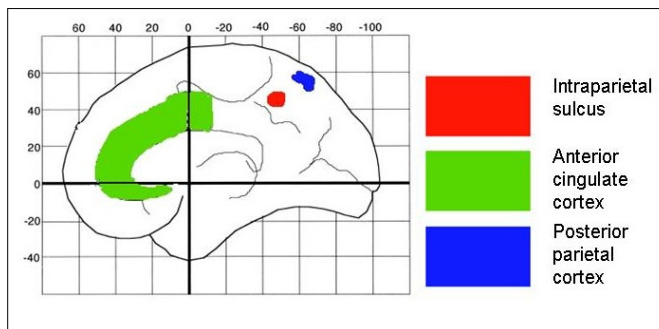


Fig. 2. brain areas involved in solving equations

### 3. Cultural transmission in context

The cumulative nature of human cultural evolution stands in stark contrast with that of non-human animals. Among other things, human cognition is characterized by an *active externalism*: humans lean heavily on environmental supports, such as the nautical slide ruler, the use of counting rods in ancient Chinese mathematics, and of course, books and other means of external storage (Clark & Chalmers 1998). Clearly, the use of external media enables us to accumulate information beyond the scope of the individual memory. Working with external symbolic media may constitute an epistemic act in itself. Some actions performed by use of external media solve problems more easily and reliably than if they would be solved in the mind alone. Take the computer game Tetris: physically rotating the two-dimensional blocks by means of a keyboard in order to fit them in the slots proves far more efficient than mental rotation (Kirsh & Maglio 1994). The use of external media makes computational solutions possible that could not have been reached without them. In mathematics, a positional system such as the Hindu numerals renders multiplications with large numbers easy and transparent, while multiplication in non-positional systems such as the Roman numerals is a far more daunting task. As I shall demonstrate, the absence or presence of symbols that represent operations influences the degree of abstraction within specific mathematical systems.

#### 3.1. Chinese algebra

Chinese algebra first blooms around the beginning of the Common Era with the anonymous *Nine Chapters on the Mathematical Arts* (*Chiu Chang Suan Shu*), which contains methods to solve simultaneous equations. It is organized around practical problems of public services, such as taxation and digging canals (Joseph 1990: 135). During the Tang dynasty (618-906), China exhibited an unusual openness to foreign influences, welcoming Arab and Indian scholars. Thanks to this influx, by the end of the Sung dynasty (900-1279), Chinese algebra had attained a level that would only be surpassed in Europe in the course of the eighteenth century (Joseph 1990). During the Ming dynasty (1368-1644), as a reaction to the Mongol

invasions, indigenous culture and values were restored. This led to a stagnation and decline in all sciences. Finally, during the Qing dynasty (1644-1911) there was an increase in influence from European mathematics, which led to the demise of Chinese algebra and other mathematical systems (Restivo 1992).

Why did Chinese algebra come to a standstill, and why was it eventually abandoned in favour of Western algebra? A possible factor is the notation system (Fig. 3). A notation system provides a way to externalize thought processes. To represent a system of equations, counting rods were arranged in such a way that each column was assigned to each equation, and each row corresponded to the coefficients of each unknown. This in turn promoted the invention of matrix methods to solve simultaneous linear equations and higher-order equations. However, these rods were less useful to express general abstract rules other than actual calculations, which preserved the concreteness of Chinese mathematics (Joseph 1990). Next to this, the venerable status of the *Nine Chapters on the Mathematical Arts* impeded further progress. Numerous mathematicians wrote commentaries on it, and as time went by commentaries on those commentaries, and even those who produced original work felt obliged to extensively refer to it. The high status of mathematicians gradually eroded as mathematics came to be perceived as a diligent and unquestioning application of ancient wisdom, rather than a creative ongoing process (Joseph 1990). Chinese mathematics never became an autonomous discipline: mathematicians remained technical experts dealing with chronology, finances, taxation, architecture, and the military (Restivo 1992).

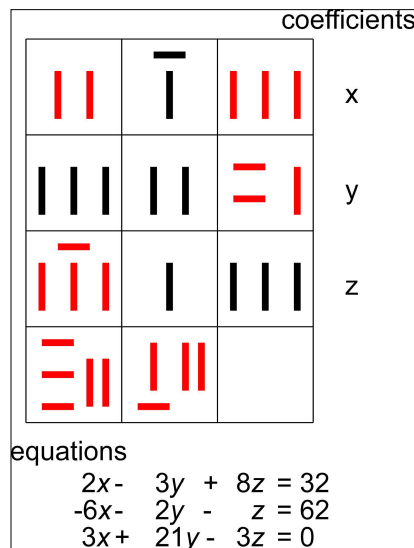


Fig. 3. simultaneous equations with Chinese counting rods (adapted from Joseph 1990, 146)

### 3.2. Islamic algebra

We now turn to the second case-study, Islamic algebra, which developed as a combination of and elaboration upon cultural inventions borrowed from other traditions, notably Greek geometry, Indian numerals and arithmetic, and Nestorian astronomy. This was possible because of the patronage of the Abbasid caliphate (758-1258), centred in Baghdad. It commissioned translations of major works by Aristotle, Galen, and many Persian and Indian scholars into Arabic. This knowledge became appropriated and reconstructed in Islamic mathematics (Berggen 1997). It built upon a combination of Greek geometric visualization and ‘proofs’, and Indian and Chinese arithmetic rules to solve equations quickly and efficiently. After the 14<sup>th</sup> century, Islamic mathematics and other sciences apparently stagnated and declined.

A plausible cause for this decline was the growing incompatibility between Islamic metaphysics and science. According to the Qu’ran, the world is governed by Allah’s uninterrupted control of all events. This line of thought inhibits the possibility of a world governed by rational, knowable and coherent laws of nature, being a principle of Greek philosophy. Nevertheless, between the 8<sup>th</sup> and 11<sup>th</sup> century, Islam did adopt scientific thought in the form of philosophy, astronomy, medicine and mathematics. However, influential theologians such as al-Ash’ari (873 - 935) and al-Ghazzali (1058 - 1111) questioned the validity of Greek philosophy which they saw as a menace to religious orthodoxy. They

became committed to Islamic occasionalism. According to this doctrine, the basic building blocks of nature are individual indivisible ‘atoms’, each of which lasts only a moment and then disappears. Allah creates the world anew each moment of time, thereby giving it pattern and persistence. The world by itself cannot exist without his continuous creation and is therefore not ruled by laws of nature. Such a worldview is deeply incompatible with the basic assumptions of science that the world is governed by rational, coherent and orderly laws (Huff 2000). Mathematics, and its application in astronomy and other branches of science did not escape this devastating attack. The 14<sup>th</sup> century theologian al-Iji (cited in Berggen 1997: 428) wrote that we can disregard mathematical entities as they are more fragile than a spider’s web. From the 10<sup>th</sup> century onwards, occasionalism became the dominant orthodoxy - and incidentally it still is. Hampered by religious orthodoxy, mathematics and other sciences could no longer flourish, with the exception of some applied mathematics that had highly political or religious importance, such as the construction of a round cupola on the square basis of mausoleums for political rulers, or the calculation of the relative position of Mecca (Berggen 1997).

### 3.3. European algebra

Our third case-study, European algebra, starts in the late 13<sup>th</sup> century commercial cities of the Catalonian, Provençal and northern Italian regions with mathematicians with a background in the teaching of commercial calculation methods. However, while showing substantial influence from Islamic mathematics, this algebra went beyond it in its ambition to solve irreducible higher-degree problems, starting with Jacopo da Firenze’s *Tractatus algorismi* (1307) (Hoyrup *in press*). Since the 17<sup>th</sup> century, mathematicians began to work at increasingly abstract problems, including algebraic series and the calculus, culminating in collective conflicts among synthesizing schools with rival programs on the foundations and nature of mathematics (Restivo 1992).

A disparate number of cultural factors have jointly contributed to this evolution. In contrast to Islamic metaphysics, Christian metaphysics endorsed the notions of natural laws and causality. The Scholastics saw evidence of God’s harmonious creation, fully in line with Plato’s system of nature, as a system of causal necessity. The metaphor of this *machina mundi* is ubiquitous in their writings. The arrival of Arabic translations of Aristotle in the 13<sup>th</sup> century gave this enthusiasm for naturalistic enquiry another powerful boost. Medieval universities institutionalized philosophical inquiry, making it acceptable that knowledge acquired through scientific investigation, became a valid source of knowledge next to the Scriptures (Huff 2000). Nevertheless, it was not until the 17<sup>th</sup> century that a fundamental shift in the perception of science enabled turnovers such as the transformation from Ptolemaic to Keplerian astronomy. While medieval writers perceived mathematics and science as an unchanging body of knowledge with deep roots in classical Greece, scholars gradually realized that mathematicians could improve the existing body of knowledge. In this newly emerged paradigm, science was no longer perceived as ancient revealed knowledge, but as a human endeavour influenced by culture and circumstance. John Wallis, a 17<sup>th</sup> century historian of mathematics, not only wrote that the Muslims had developed and improved classical texts, but proudly stated the further achievements of his contemporaries (Stedall 2001).

European mathematicians thus became increasingly aware that they themselves created the levels of abstraction in mathematics. This realization led to many innovations such as the introduction of imaginary numbers. As noted before, externalisation of knowledge constitutes an epistemic action. Take  $i$ , which symbolizes an operation that cannot actually be performed – namely extracting a square root from a negative number. By

denoting this operation it becomes possible to integrate roots from negative numbers in operations, thus vastly extending the scope of mathematics. Only European mathematics has consistently aimed at representing such opaque operations by a constant set of symbols (Restivo 1992). Indian mathematicians, for example, used the symbol '0' both for zero and for the unknown in equations (Joseph 1990). Such ambiguities, or in the case of Chinese algebra, a lack of external symbols for operations, inevitably limit the degree of abstraction in mathematics.

#### 4. Conclusion

In conclusion, I have argued that algebra is a cognitive system with emergent properties, resulting from the co-optation of evolved mental abilities and the externalisation of operations in a symbolic system. This system needs extensive cultural support, both from the metaphysics of the society and the ontological status of mathematics. It is crucial for mathematics and other sciences that a metaphysical view is endorsed in which the world is governed by causal laws, and in which humans are capable of acquiring reliable knowledge of the world. Once mathematicians realized their practice is a human endeavour, and that they can improve it, they could create higher levels of abstraction by symbolically representing operations. Early modern European mathematics was not superior to its Chinese and Arabian contemporaries - rather, it could develop because cultural conditions were more favourable.

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